

COMMUTATOR WIDTH OF CHEVALLEY GROUPS OVER RINGS OF STABLE RANK 1

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ABSTRACT. An estimate on the commutator width is given for Chevalley groups over rings of stable rank 1, and the general method suitable for other rings of small dimension.

1. INTRODUCTION

The study of commutators in linear groups over fields has a rich history, culminating in the celebrated proof of Ore conjecture [EG98, LOST10], while the groups over rings received much less attention. It was shown in [VW90], that for an associative ring R of stable rank 1 the group $GL(n, R)$, $n \geq 3$ has the commutator width ≤ 2 , i.e. that every element of its commutator subgroup can be written as a product of two commutators. It was then generalised (with somewhat worse bounds) in [AVY95] to symplectic, orthogonal and unitary groups in even dimension in the context of hyperbolic unitary groups [BV00]. The goal of the present paper is to provide a similar result for exceptional groups.

The proof follows the line of those in [VW90, AVY95], but tries to avoid explicit matrix calculations, thus giving a simpler and (almost) uniform treatment for Chevalley groups of all normal types. Apart from exceptional groups, it also covers Spin and odd-dimensional orthogonal groups, which were not considered in the previous papers.

It was a surprise for the author that the case of special linear group over rings of stable rank 1 is not presented in [VW90] or anywhere else. Apparently, the reason is that one has to do some additional considerations as in Lemma 5 and Remarks 3 and 4 (compare with Proposition 8

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of [VW90]), leading to certain (insignificant) complications in the proof of Theorem 1 for SL_{4k+2} .

Let Φ be a reduced irreducible root system, $\alpha_1, \dots, \alpha_\ell$ its fundamental roots (numbered as in Bourbaki), $W(\Phi)$ the corresponding Weyl group, generated by the simple reflections $\sigma_1, \dots, \sigma_\ell$. For a commutative ring R with 1 by the Chevalley group $G(\Phi, R)$ we mean the group of point of the correspong Chevalley-Demazure group scheme $G(\Phi, -)$. Unless specified otherwise, all groups are assumed to be simply connected.

We extensively use the weight diagrams, see [PSV98, Vav00, Vav01]. Weight diagrams allow to visualize the action of the elementary root unipotents $x_\alpha(t)$ (the generators of the elementary subgroup $E(\Phi, R)$). Below is the weight diagram for the natural vector representation of $SL_{\ell+1}$, the nodes correspond to the weights of the representation, and the edges to the fundamental roots. $x_{ij}(t)$ acts on $(v_k)_{k=1}^{\ell+1}$ by adding tv_j to v_i , or, in terms of the diagram, along the chain connecting j to i .

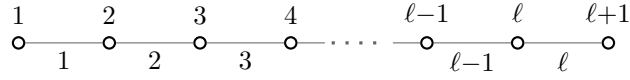


FIGURE 1. (A_ℓ, ϖ_1)

We call a set of roots $S \subseteq \Phi$ closed if for any $\alpha, \beta \in S$, if $\alpha + \beta \in \Phi$ is a root, then $\alpha + \beta \in S$. Two important examples of closed sets of roots are the following. Let $m_i(\alpha)$, $i = 1, \dots, \ell$ denote the coefficients in the expansion of α as an integer linear combination of the fundamental roots, then put

$$\Sigma_k = \{\alpha \in \Phi \mid m_k(\alpha) \geq 1\}, \quad \Delta_k = \{\alpha \in \Phi \mid m_l(\alpha) = 0\}.$$

The sets Σ_k are unipotent (i.e. $S \cap -S = \emptyset$), and Δ_k are symmetric (i.e. $S = -S$). The notation $\Sigma_k^=n$, $\Sigma_k^{\leq n}$ and $\Sigma_k^{\geq n}$ is self-explaining.

The sets of all positive and negative roots Φ^+, Φ^- are also both closed and unipotent.

To a closed set of roots S we associate a subgroup $E(S, R) = \langle x_\alpha(t) \mid \alpha \in S, t \in R \rangle$. The ring R is often clear from the context and thus omitted in the notation. If S is unipotent, we sometimes write $U(S)$

instead of $E(S)$. The unitriangular subgroups $U(\Phi^\pm)$ are denoted by U^\pm .

$U(\Sigma_k)$ is the unipotent radical of the corresponding parabolic subgroup P_k (or $E(\Delta_k \cup \Sigma_k)$). Levi decomposition states that $E(\Delta_k \cup \Sigma_k)$ is the semi-direct product of its elementary Levi subgroup $E(\Delta_k)$ and its normal subgroup $U(\Sigma_k)$.

2. COMMUTATORS AND COMPANION MATRICES

Definition 1. Fix an element $w \in W(\Phi)$ and a natural number n and set

$$\begin{aligned}\Omega_n^w &= \{\alpha \in \Phi^+ \mid w^{n+1}\alpha \in \Phi^-, w\alpha, w^2\alpha, \dots, w^n\alpha \in \Phi^+\}, \\ \Theta^w &= \{\alpha \in \Phi^+ \mid w^k\alpha \in \Phi^+ \forall k \in \mathbb{Z}\}.\end{aligned}$$

When the choice of particular element w is clear from the context, we usually omit the super index and simply write Θ and Ω_n .

Note that $\Phi^+ = \Theta \cup \bigcup_{k \geq 0} \Omega_k$ and the union is disjoint.

Remark 1. Θ is closed for any $w \in W$.

Proof. Suppose there are $\alpha, \beta \in \Theta$ with $\alpha + \beta \in \Phi^+ \setminus \Theta$. Then there is some $k \geq 0$ such that $w^k(\alpha + \beta) \in \Phi^-$. So either $w^k\alpha$ or $w^k\beta$ is negative. \square

Lemma 1. $\Theta \cup \bigcup_{k=0}^n \Omega_k$ is closed for any n . As a corollary, $\Phi^+ \setminus \bigcup_{k \geq n} \Omega_k$ is closed for any n .

Proof. Suppose there are $\alpha, \beta \in \Theta \cup \bigcup_{k=0}^n \Omega_k$ such that $\alpha + \beta \in \bigcup_{k > n} \Omega_k$. Then there exists $m > n$ with $w^{m+1}(\alpha + \beta) \in \Phi^-$ and $w^i(\alpha + \beta) \in \Phi^+$ for all $i = 0, \dots, m$.

Thus either $w^{m+1}\alpha$ or $w^{m+1}\beta$ is negative, so one of α, β lies outside of Θ . Since $w^{n+1}(\alpha + \beta) \in \Phi^+$, one has $\alpha \in \Theta \cup \bigcup_{k > n} \Omega_k$ or $\beta \in \Theta \cup \bigcup_{k > n} \Omega_k$. If $\alpha \in \Theta$, then $\beta \in \bigcup_{k > n} \Omega_k$, a contradiction, and similarly for $\beta \in \Theta$. \square

We write, as usual

$$w_\alpha(u) = x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u), \quad h_\alpha(u) = w_\alpha(u)w_\alpha(-1).$$

Remark 2. For any $\alpha, \beta \in \Phi$, $t \in R$, $u, v \in R^*$

$$\begin{aligned} w_\alpha(u)x_\beta(t)w_\alpha(u)^{-1} &= x_{\sigma_\alpha\beta}(\pm u^{-\langle\beta,\alpha\rangle}t), \\ w_\alpha(u)w_\beta(v)w_\alpha(u)^{-1} &= w_{\sigma_\alpha\beta}(\pm u^{-\langle\beta,\alpha\rangle}v), \\ h_\alpha(u)w_\beta(v)h_\alpha(u)^{-1} &= w_\beta(u^{\langle\beta,\alpha\rangle}v). \end{aligned}$$

The above relations hold on the level of Steinberg group, while on the level of elementary group they imply the following formula (note that signs cancel out):

$$w_\alpha(1)h_\beta(\varepsilon)w_\alpha(1)^{-1} = h_{\sigma_\alpha\beta}(\pm\varepsilon)h_{\sigma_\alpha\beta}(\pm 1)^{-1} = h_{\sigma_\alpha\beta}(\varepsilon).$$

The extended Weyl group $\widetilde{W}(\Phi)$ is the subgroup of $G(\Phi, R)$, generated by $w_\alpha(1)$, $\alpha \in \Phi$. If $2 \neq 0$ in R , it coincides with the $N(\Phi, \mathbb{Z})$, the group of integer point of the torus normalizer. It is an extension $C_2^\ell \hookrightarrow N(\Phi, \mathbb{Z}) \twoheadrightarrow W(\Phi)$, and the action of the generators on the kernel is described by the above formula.

Definition 2. Denote by $\widetilde{\pi}$ the following element of the Weyl group:

- A_ℓ, B_ℓ, C_ℓ : $\widetilde{\pi} = \sigma_1\sigma_2 \dots \sigma_\ell$, a Coxeter element;
- D_ℓ : $\widetilde{\pi} = \sigma_\ell \dots \sigma_2\sigma_1$;
- E_6 : $\widetilde{\pi} = \sigma_1\sigma_3\sigma_4\sigma_5\sigma_6$, a Coxeter element of an A_5 subsystem;
- E_7 : $\widetilde{\pi} = \sigma_1\sigma_3\sigma_2\sigma_4\sigma_5\sigma_6\sigma_7$;
- E_8 : $\widetilde{\pi} = \sigma_1\sigma_3\sigma_2\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8$;
- F_4 : $\widetilde{\pi} = \sigma_1\sigma_2\sigma_3\sigma_4$;
- G_2 : $\widetilde{\pi} = \sigma_2\sigma_1$.

By π denote a lift of $\widetilde{\pi}$ to the extended Weyl group, obtained by sending σ_i to $w_i(1)$.

For any Coxeter element w_c one has $\Theta^{w_c} = \emptyset$. Thus with the choice of $\widetilde{\pi}$ as above $\Theta^{\widetilde{\pi}} = \emptyset$ in all cases except E_6 , when $\Theta^{\widetilde{\pi}} = \Sigma_2$.

Definition 3. A companion matrix is an element of the form $u\pi$ with $u \in U(\Sigma)$, where $\Sigma = \Omega_0^{\widetilde{\pi}}$ for $\Phi \neq E_6$ and $\Sigma = (\Theta^{\widetilde{\pi}} \setminus \{\alpha_2\}) \cup \Omega_0^{\widetilde{\pi}}$ for $\Phi = E_6$. Depending on the root system it can be described as:

- A_ℓ : $\Sigma = \Sigma_\ell$;
- B_ℓ : $\Sigma = (\Sigma_\ell^{\neq 2} \cap \Sigma_{\ell-1}^{\neq 1}) \cup \{\alpha_\ell\}$ (marked black on the weight diagram of the adjoint representation, see Figure 3);
- C_ℓ : $\Sigma = (\Sigma_\ell^{\neq 1} \cap \Sigma_{\ell-1}^{\neq 1}) \cup \{\alpha_\ell\} = \Sigma_\ell^{\neq 1} \cap \Sigma_{\ell-1}^{\leq 1}$ (Figure 2);

- D_ℓ : $\Sigma = \Sigma_1 \cap (\Delta_\ell \cup \Delta_{\ell-1})$ (Figure 4);
- E_6 : $\Sigma = (\Sigma_6 \cap \Delta_2) \cup (\Sigma_2 \setminus \{\alpha_2\})$ (Figure 9);
- E_7, E_8, F_4, G_2 : see Figures 5, 6, 7, 8.

The above description (for $\Phi \neq E_6$) is obtained as follows: first, one checks that $\tilde{\pi}$ sends the right hand side to Φ^- and that the number of roots in it equals the rank of Φ . Then it remains to note that $|\Omega_0^{\tilde{\pi}}| = \text{rk}(\Phi)$. This follows from the fact that all orbits of a Coxeter element w_c have the same size, equal to the Coxeter number h (since w_c acts by rotation by $2\pi/h$ on its Coxeter plane and no root projects to zero) and that $|\Phi| = h \cdot \text{rk}(\Phi)$. For $\Phi = E_6$ one applies this argument to the A_5 -subsystem Δ_2 .

Lemma 2. *For any $u \in U^+(\Phi)$ exists $\eta \in U^+(\Phi)$ such that $\eta u \pi \eta^{-1}$ is a companion matrix.*

Proof. Consider Ω_k for $w = \tilde{\pi}$ (see Definition 1) and denote by N the maximal natural number such that $\Omega_N \neq \emptyset$.

Write u as a product θv , where $\theta = \prod_{\alpha \in \Omega_N} x_\alpha(c_\alpha)$ and $v \in E(\Phi^+ \setminus \Omega_N)$. Consider the conjugate $\theta^{-1} u \pi \theta$. It follows from Remark 2 that for $\alpha \in \Omega_N$ $\pi x_\alpha(c_\alpha) \pi^{-1} \in E(\Omega_{N-1}) \subset E(\Phi^+ \setminus \Omega_N)$, and thus $\pi \theta \in E(\Phi^+ \setminus \Omega_N) \pi$, so $\theta^{-1} u \pi \theta = u' \pi$ for some $u' \in E(\Phi^+ \setminus \Omega_N)$, since the latter set of roots is closed.

Now we can rewrite u' as a product $\theta' v'$, where $\theta' = \prod_{\alpha \in \Omega_{N-1}} x_\alpha(c_\alpha)$ and $v' \in E(\Phi^+ \setminus (\Omega_N \cup \Omega_{N-1}))$. Repeat the previous step to get an element of the form $u'' \pi$ with $u'' \in E(\Phi^+ \setminus (\Omega_N \cup \Omega_{N-1}))$.

Repeating this procedure $N - 1$ times, we eventually get an element of the form $u \pi$ with $u \in E(\Phi^+ \setminus \bigcup_{k>0} \Omega_k) = E(\Theta \cup \Omega_0)$.

Since $\Theta \cup \Omega_0$ coincides with Σ in all cases except E_6 , we are almost done. For $\Phi = E_6$ one additionally has to eliminate $x_{\alpha_2}(\ast)$ in the same way (which gives the closed set Σ). \square

A lift w of a Coxeter element to the extended Weyl group is essentially a set of signs $s_i^w = \pm 1$, assigned to the fundamental roots α_i .

Lemma 3. *Let $\Phi \neq A_\ell, D_\ell, E_7$ and w_1, w_2 be two lifts of a single Coxeter element to the extended Weyl group $N(\Phi, \mathbb{Z})$. Then w_1 and w_2 are conjugated under the action of $H(\Phi, \mathbb{Z})$.*

Proof. The action of an element $h \in H(\mathbb{Z})$ changes some of s_i^w , while leaving the others intact. It follows from Remark 2 that $h_\alpha(-1)$ only changes signs assigned to the roots β with odd $\langle \beta, \alpha \rangle$. The latter can be easily computed, for example, as follows: if $\beta - r\alpha, \dots, \beta + q\alpha$ is the α -series through β , then $\langle \beta, \alpha \rangle = r - q$.

To find a suitable element of $H(\mathbb{Z})$ we have to do some case-by-case analysis. In each case we provide a procedure for transforming one set of signs (w_1) into another (w_2), that is a chain of “elementary” transformations $w \mapsto h_\alpha(\pm 1)w$. The signs of the current value of w will be denoted simply by s_i .

$\Phi = \mathbf{B}_\ell$, $\ell \geq 3$: we start with obtaining the desired value of s_ℓ by conjugating w with $h_{\alpha_{\ell-1}}(\pm 1)$. Now we take $\gamma_1 = \alpha_1 + \alpha_2 + \dots + \alpha_\ell$ and note that $\langle \alpha_1, \gamma_1 \rangle = 1$ and $\langle \alpha_\ell, \gamma_1 \rangle = 0$. This shows that conjugating with h_{γ_1} allows us to change s_1 while not changing s_ℓ . Analogously, we can take $\gamma_k = \alpha_{k-1} + 2\alpha_k + \dots + 2\alpha_\ell$ to change s_k , $k = \ell - 1, \ell - 2, \dots, 3$. Each of h_{γ_k} affects only s_k and s_{k-1} , and the latter is fixed by $h_{\gamma_{k-1}}$. The last step is to take $\gamma_2 = \alpha_{\max} = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_\ell$, for h_{γ_2} changes only s_2 .

$\Phi = \mathbf{C}_\ell$, $\ell \geq 2$: we start with fixing s_ℓ by conjugating w with h_γ for $\gamma = \alpha_{\ell-1} + \alpha_\ell$ (indeed, $\langle \alpha_\ell, \gamma \rangle = 1$). Now we change s_k , $k = 1, 2, \dots, \ell - 1$ by using $h_{\alpha_{k+1}}$.

$\Phi = \mathbf{E}_6$: first change s_2 by conjugating with h_{α_4} , then use α_1 and α_6 for s_3 and s_5 , then α_3 and α_5 for s_1 and s_6 , and finally α_2 to change s_4 .

$\Phi = \mathbf{E}_8$: use α_3 to change s_1 , then use $\alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8$ to change s_2, s_4, s_5, s_6, s_7 and finish by using α_1 for s_3 and α_{\max} for s_8 .

$\Phi = \mathbf{F}_4$: use α_1 and α_2 for s_2 and s_1 , then α_3 and α_4 for s_4 and s_3 .

$\Phi = \mathbf{G}_2$: use α_1 and α_2 for s_2 and s_1 . \square

The failure of the above lemma for \mathbf{E}_7 is amusing, yet the reason is unclear.

Since for $\Phi = \mathbf{A}_\ell, \mathbf{D}_\ell, \mathbf{E}_7$ we can't use Lemma 3, we have to do some additional calculations in these cases.

Let w_0 denote the longest element of the Weyl group. We write \widehat{w}_0 for its obvious lift to the extended Weyl group, obtained by sending each of σ_i in the reduced expression to $w_i(1)$. We will later fix another lift in case $\Phi = \mathbf{A}_\ell$.

Lemma 4. *For $\Phi = A_\ell, D_\ell, E_7$ one has $\widehat{w}_0 w_i(1) \widehat{w}_0^{-1} = w_j(1)$, where $\alpha_j = -w_0(\alpha_i)$.*

Proof. For A_ℓ and D_ℓ this can be done by explicit matrix calculation (this is done for orthogonal group in [AVY95] and immediately follows for Spin group, since the central factor doesn't play any role).

For E_7 it is not a good idea to write down matrices, but one can do something very similar. Namely, after choosing a positive basis for the microweight representation (E_7, ϖ_7) one has very simple and explicit description of the action of $\widetilde{W}(\Phi)$.

Let Λ denote the set of weights, α a fundamental root, then

$$w_\alpha(1)v^\lambda = \begin{cases} v^\lambda, & \text{if } \lambda \pm \alpha \notin \Lambda, \\ v^{\lambda+\alpha}, & \text{if } \lambda + \alpha \in \Lambda, \\ -v^{\lambda-\alpha}, & \text{if } \lambda - \alpha \in \Lambda. \end{cases}$$

Now elements of $\widetilde{W}(\Phi)$ act by signed permutation on Λ , so it is a routine to check that $\widehat{w}_0 w_i(1) = w_i(1) \widehat{w}_0$ for each i . It is, however, much less amusing, so the author advises to put it into a computer instead. \square

As a corollary, we see that in case $\Phi = D_\ell$ one has $\widehat{w}_0 \pi \widehat{w}_0 = \pi$, since $w_\ell(1)$ and $w_{\ell-1}(1)$ commute.

Let p_n denote the $n \times n$ peridentity matrix, that is $p_n = (\delta_{i,n-j})_{i,j=1,\dots,n}$. If $n \neq 4k+2$, then either $\det(p_n) = 1$ or $\det(-p_n) = 1$, so for $SL(n, R) = G(A_{n-1}, R)$ we specifically fix \widehat{w}_0 to be p_n or $-p_n$, which is not the obvious lift of w_0 . But with this choice one has

$$\widehat{w}_0 \cdot w_\ell(-1) \dots w_1(-1) \cdot \widehat{w}_0^{-1} = w_1(1) \dots w_\ell(1),$$

which can be written simply as $\widehat{w}_0 \pi^{-1} \widehat{w}_0^{-1} = \pi$.

For $n = 4k+2$ neither p_n nor $-p_n$ lies in $SL(n, R)$. In this case we set \widehat{w}_0 to be an anti-diagonal matrix with 1's and -1 's alternating. Then $\widehat{w}_0 \pi^{-1} \widehat{w}_0^{-1} = -\pi$.

Lemma 5. *Let $\Phi = A_\ell$, $\ell \neq 4k+1$ or $\Phi = E_6$. If x is similar to a companion matrix, then so is x^{-1} .*

Proof. Write $\eta x \eta^{-1} = u \pi$ for some $u \in U(\Sigma)$, so $\eta x^{-1} \eta^{-1} = \pi^{-1} u^{-1}$.

In case $\Phi = A_\ell$, $\ell \neq 4k+1$ conjugate it with \widehat{w}_0 to get $\pi u'$, where $u' \in U(w_0\Sigma)$. Note that $w_0\Sigma_\ell = -\Sigma_1 = \widetilde{\pi}\Sigma_\ell$, thus $\pi u' = u''\pi$ for some $u'' \in U(\Sigma_\ell)$.

In case $\Phi = E_6$ take v_0 to be the longest element of Δ_2 and \widehat{v}_0 its lift to the torus normalizer. Conjugate $\pi^{-1}u^{-1}$ with \widehat{v}_0 to get $\rho u'$, where $u' \in U(v_0\Sigma)$ and ρ is a (probably different) lift of $\widetilde{\pi}$, and with ρ^{-1} to get $u'\rho$. Conjugating it, if necessary, with a suitable element of $H(\mathbb{Z})$ (see Lemma 3), we can assume $\rho = \pi$. Since $v_0(\Sigma_6 \cap \Delta_2) = \widetilde{\pi}(\Sigma_6 \cap \Delta_2) = -\Sigma_1 \cap \Delta_2$ and $v_0\Sigma_2 = \widetilde{\pi}\Sigma_2 = \Sigma_2$, one has $u'\pi = \pi u''$ for $u'' \in U(\Sigma \cup \{\alpha_2\})$, which is conjugated with $u''\pi$. It remains to conjugate it with $x_{\alpha_2}(\ast)$ as in the proof of Lemma 2. \square

Remark 3. Let $\Phi = A_{4k+1}$. If x is similar to a companion matrix, then x^{-1} is similar to a minus companion matrix.

Proof. Repeat the proof of Lemma 5, now using $\widehat{w}_0\pi^{-1}\widehat{w}_0^{-1} = -\pi$. \square

Lemma 6. Let $\Phi \neq A_{4k+1}$. For any $v \in U^-(\Phi)$ exists $\eta \in E(\Phi)$ such that $\eta v \pi \eta^{-1}$ is a companion matrix.

Proof. Note that $\widehat{w}_0 v \widehat{w}_0^{-1} \in U^+(\Phi)$.

In case $\Phi = B_\ell, C_\ell, D_\ell, E_7, E_8, F_4, G_2$ one has $w_0\widetilde{\pi}w_0 = \widetilde{\pi}$ and thus $\widehat{w}_0\pi\widehat{w}_0^{-1} = \rho$ for some lift ρ . This lift is either equal to π (in cases D_ℓ, E_7 by Lemma 4) or can be transformed to π by conjugating with an element of $H(\mathbb{Z})$ (in all other cases by Lemma 3). Then one applies Lemma 2.

If $\Phi = A_\ell$, $\ell \neq 4k+1$ or $\Phi = E_6$, the longest element sends $\widetilde{\pi}$ to its inverse, so $\widehat{w}_0\pi\widehat{w}_0^{-1} = \rho^{-1}$. Thus $\widehat{w}_0 v \pi \widehat{w}_0^{-1} \in U^+(\Phi)\rho^{-1}$ and

$$\rho^{-1}\widehat{w}_0 v \pi \widehat{w}_0^{-1} \rho \in \rho^{-1}U^+(\Phi),$$

which is by Lemma 5 similar to a companion matrix as the inverse of an element from $U^+(\Phi)\rho$ (in case $\Phi = A_\ell$ one can assume $\rho = \pi$ as in the proof of Lemma 5, while in case $\Phi = E_6$ one applies Lemma 3). \square

Remark 4. Let $\Phi = A_{4k+1}$. For any $v \in U^-(\Phi)$ exists $\eta \in E(\Phi)$ such that $\eta v \pi \eta^{-1}$ is a minus companion matrix.

Proof. Repeat the proof of Lemma 6, using Remark 3 instead of Lemma 5. \square

3. PROOF OF THE MAIN RESULT

Lemma 7. *For the vector representation $(\mathbf{A}_\ell, \varpi_1)$ there exists an element $g \in E(\ell + 1, R)$, such that $g - 1 \in E(\ell + 1, R)$.*

Proof. For $\ell = 1$ and $\ell = 2$ such elements are delivered by the matrices

$$g_1 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and for arbitrary ℓ one simply composes them into a block diagonal matrix. \square

Lemma 8. *If $\text{rk } \Phi \geq 2$ and $\Phi \neq \mathbf{C}_\ell$, then every element $\theta \in U(\Sigma)$ is a product of at most N commutators, where*

- $N = 1$ in case $\Phi = \mathbf{A}_\ell, \mathbf{F}_4, \mathbf{G}_2$;
- $N = 2$ in case $\Phi = \mathbf{B}_\ell, \mathbf{C}_\ell, \mathbf{D}_\ell, \mathbf{E}_7, \mathbf{E}_8$;
- $N = 3$ in case $\Phi = \mathbf{E}_6$.

Proof. We start with working out the case $\Phi = \mathbf{A}_\ell$. Denote $\Delta = \Delta_\ell$, then the Levi factor $E(\Delta)$ acts on the unipotent radical $U(\Sigma_\ell)$.

Write $\theta = \prod_{\alpha \in \Sigma} x_\alpha(\xi_\alpha)$. An element $g \in E(\Delta) = E(\ell, R)$ acts on the vector consisting of ξ_α exactly as in $(\mathbf{A}_{\ell-1}, \varpi_1)$. To avoid confusion the result will be denoted by ${}^g(\xi_\alpha)$.

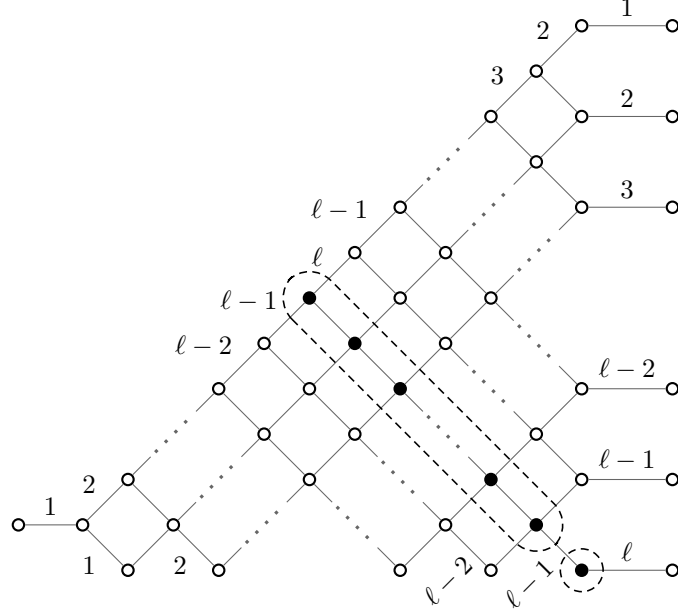
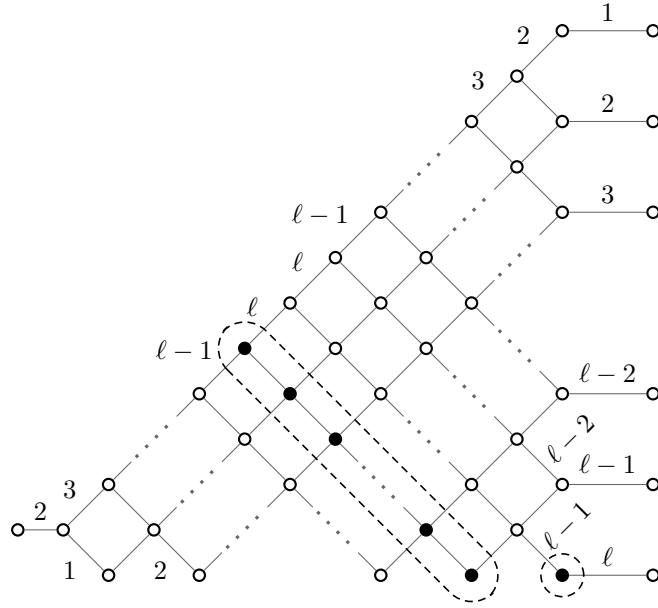
Let $g \in E(\Delta)$ be the element, constructed in Lemma 7. Then one has

$$\begin{aligned} \eta &= \left[g, \prod_{\alpha \in \Sigma} x_\alpha(\zeta_\alpha) \right], \text{ where } (\zeta_\alpha) = (g^{-1})^{-1}(\xi_\alpha). \\ g \cdot \prod_{\alpha \in \Sigma} x_\alpha(\zeta_\alpha) \cdot g^{-1} &= \prod_{\alpha \in \Sigma} x_\alpha(\zeta'_\alpha) \text{ with } (\zeta'_\alpha) = {}^g(\zeta_\alpha), \\ \eta &= \prod_{\alpha \in \Sigma} x_\alpha(\zeta''_\alpha) \text{ with } (\zeta''_\alpha) = (\zeta'_\alpha) - (\zeta_\alpha) = (g^{-1})(\zeta_\alpha) = (\xi_\alpha). \end{aligned}$$

Thus in this case $\theta = \eta$, a commutator.

If $\Phi = \mathbf{C}_\ell$, we first write

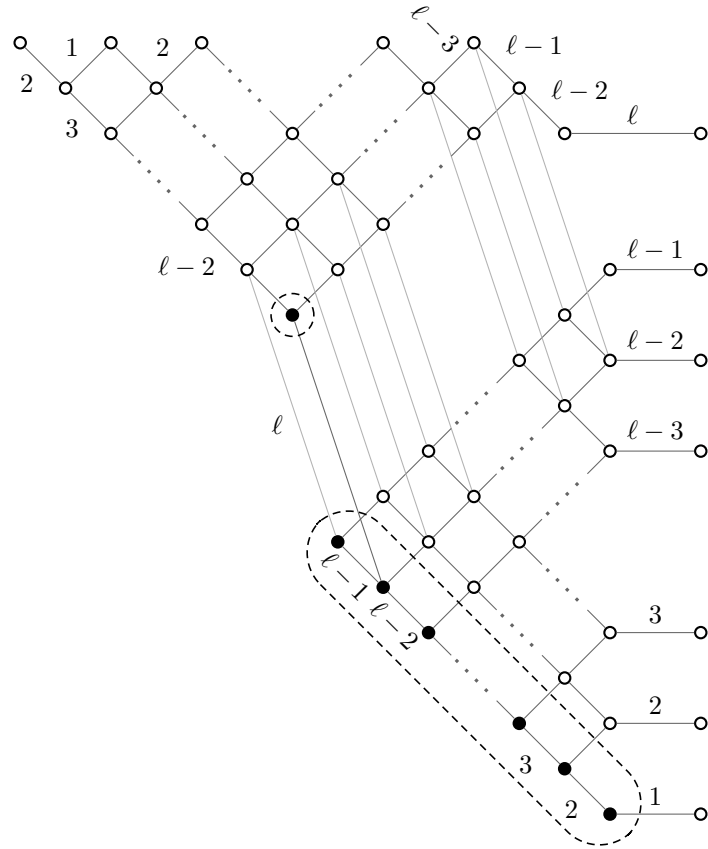
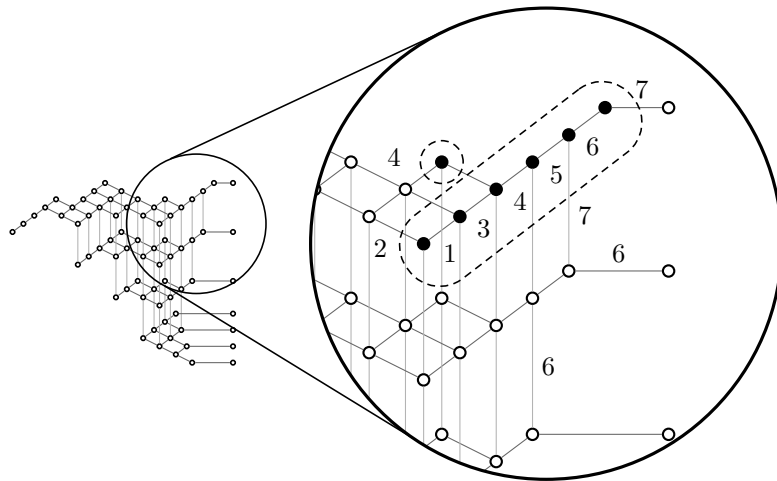
$$\begin{aligned} \theta &= x_{\alpha_\ell}(a_\ell) \cdot \prod_{\alpha \in \Sigma'} x_\alpha(a_\alpha) = x_{\alpha_\ell}(a_\ell) \cdot \theta', \text{ where } \Sigma' = \Sigma \setminus \{\alpha_\ell\}, \\ x_{\alpha_{\ell-1} + \alpha_\ell}(*) x_{\alpha_\ell}(t) &= [x_{2\alpha_{\ell-1} + \alpha_\ell}(1), x_{-\alpha_\ell}(\pm t)]. \end{aligned}$$

FIGURE 2. $(C_\ell, 2\varpi_1)$ FIGURE 3. (B_ℓ, ϖ_2)

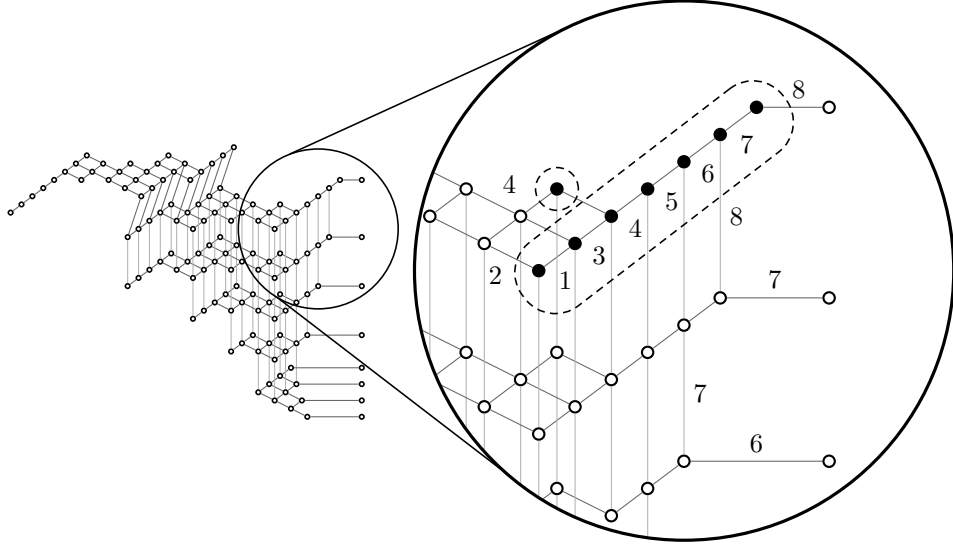
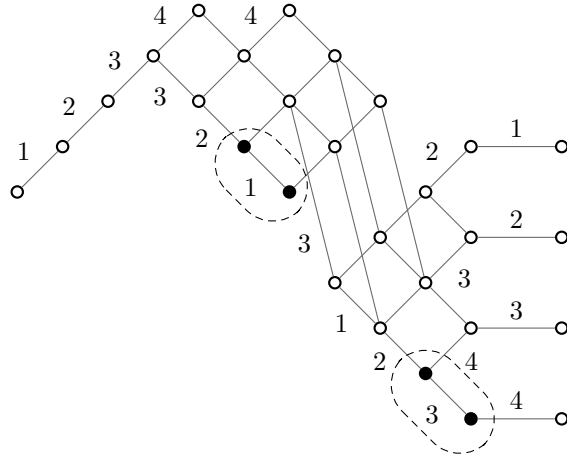
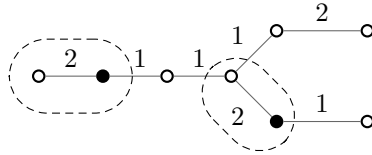
Since $\alpha_{\ell-1} + \alpha_\ell \in \Sigma = \Sigma_\ell^=1 \cap \Sigma_{\ell-1}^{\leq 1}$,

$x_\alpha(t) = c \cdot x_{\alpha_{\ell-1} + \alpha_\ell}(*),$ where c is a commutator,

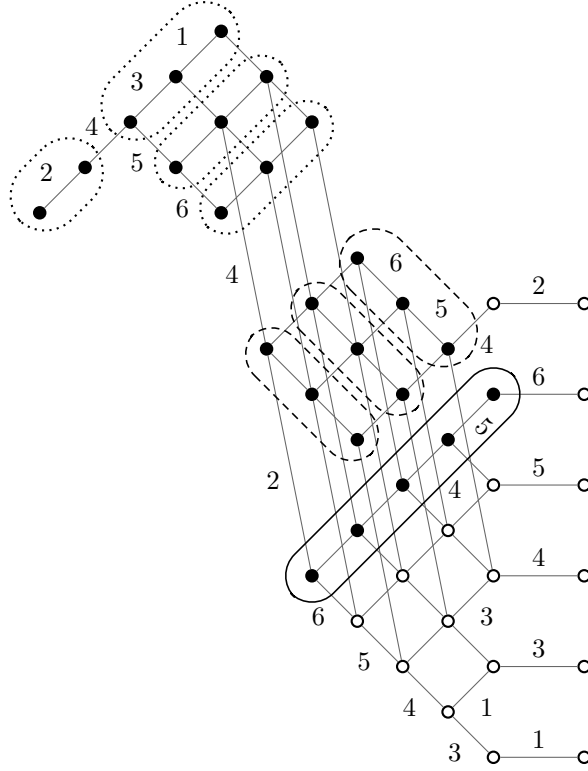
$\theta = c \cdot \theta'',$ for some $\theta'' \in U(\Sigma \setminus \{\alpha_\ell\}).$

FIGURE 4. (D_ℓ, ϖ_2) FIGURE 5. (E_7, ϖ_1)

Then $g \in E(\Delta \cap \Delta_{\ell-1})$, constructed in Lemma 7, doesn't use roots from $\Sigma_{\ell-1}$ and it is clear from the diagram (Figure 2) that it acts on

FIGURE 6. (E_8, ϖ_8) FIGURE 7. (F_4, ϖ_1) FIGURE 8. (G_2, ϖ_2)

θ'' as prescribed by Lemma 7, allowing to repeat the argument we used for $\Phi = A_\ell$. Thus θ is a product of two commutators.


 FIGURE 9. (E_6, ϖ_2)

If $\Phi = B_\ell$ (Figure 3), we do the same as for C_ℓ . This time we exclude α_ℓ and note that $\Delta_\ell \cap \Delta_{\ell-1}$ acts on the chain $\Sigma \setminus \{\alpha_\ell\}$ as $(A_{\ell-2}, \varpi_1)$. Again,

$$x_{\alpha_\ell}(t) = [x_{\alpha_{\ell-1}+\alpha_\ell}(1), x_{-\alpha_{\ell-1}}(\pm t)] \cdot x_{\alpha_{\ell-1}+2\alpha_\ell}(*).$$

If $\Phi = D_\ell$ (Figure 4), we divide Σ into two parts: $\Sigma_1 \cap \Delta_{\ell-1}$ and $\{\alpha = \alpha_1 + \dots + \alpha_{\ell-1}\}$. The first one is subject to the action of $\Delta_{\ell-1}$ (of type $A_{\ell-1}$), while $x_\alpha(t) = [x_{\alpha+\alpha_\ell}(t), x_{-\alpha_\ell}(1)]$.

The very same method works for $\Phi = E_7, E_8$, see Figures 5, 6.

In case $\Phi = F_4$ (Figure 7) the subsystem subgroup $E(\langle \alpha_1, \alpha_3 \rangle) \cong E(A_1) \times E(A_1)$ acts on $U(\Sigma)$, and the edges labeled 1 and 3 only meet each other outside Σ .

If $\Phi = G_2$ (Figure 8), we slightly extend Σ along the edges labeled 2 to some Σ' , then $E(\langle \alpha_2 \rangle)$ acts simultaneously on both chain of $U(\Sigma')$ (as a vector, it has zeroes on the additional roots).

In the remaining case $\Phi = E_6$ (Figure 9) we split Σ into three parts, marked by solid, dashed and dotted outlines in the figure, acted on by

$E(\Delta_6)$, $E(\langle \alpha_5, \alpha_6 \rangle)$ and $E(\langle \alpha_1, \alpha_2, \alpha_3 \rangle)$ (of type A_5 , A_2 and $A_2 \times A_1$) correspondingly. \square

Definition. A commutative ring R is said to have stable rank 1, if for any $a, b \in R$, such that they generate R as an ideal, there is $c \in R$, such that $a + bc \in R^*$ is invertible.

Examples of rings of stable rank 1 are fields, semilocal rings, boolean rings, the ring of all algebraic integers, the disc-algebra.

Theorem 1. Let Φ be a root system, R a commutative ring of stable rank 1. Then each element $g \in E(\Phi, R)$ is a product of at most N commutators in $E(\Phi, R)$, where

- $N = 3$ in case $\Phi = A_\ell, F_4, G_2$;
- $N = 4$ in case $\Phi = B_\ell, C_\ell, D_\ell, E_7, E_8$;
- $N = 5$ in case $\Phi = E_6$.

Proof. We will use the so-called unitriangular factorization

$$E(\Phi, R) = U^+(\Phi, R) U^-(\Phi, R) U^+(\Phi, R) U^-(\Phi, R),$$

that holds for any elementary Chevalley group over any commutative ring of stable rank 1 [VSS12].

Assume first $\Phi \neq A_{4k+1}$. Write $g = u_1 v_1 u_2 v_2$, where $u_i \in U^+$ and $v_i \in U^-$. Then $g = u_3 c_1 v_3 = c_2 u_3 v_3 = c_2 (u_3 \pi) (\pi^{-1} v_3)$, where c_i are commutators. Denote $\varphi = u_3 \pi$ and $\psi = \pi^{-1} v_3$. By Lemmas 2 and 6 there exist $\mu, \nu \in E(\Phi)$ such that $z_1 = \mu \varphi \mu^{-1}$ and $z_2 = \nu \psi^{-1} \nu^{-1}$ are companion matrices. Since $z_1 z_2^{-1} = \zeta \in U(\Sigma)$, one has

$$\mu \varphi \mu^{-1} = \zeta \nu \psi^{-1} \nu^{-1}.$$

Thus $\varphi = \mu^{-1} \zeta \nu \psi^{-1} \nu^{-1} \mu$ and

$$\begin{aligned} \varphi \psi &= \mu^{-1} \zeta \nu \psi^{-1} \nu^{-1} \mu \cdot \psi = \\ &= \mu^{-1} \zeta \nu \cdot \psi^{-1} \cdot \nu^{-1} \cdot (\zeta^{-1} \mu \cdot \psi \cdot \psi^{-1} \mu^{-1} \zeta) \cdot \mu \psi = \\ &= [\mu^{-1} \zeta \nu, \psi^{-1}] \cdot \psi^{-1} \mu^{-1} \zeta \mu \psi = [\mu^{-1} \zeta \nu, \psi^{-1}] \cdot \zeta^{\mu \psi}. \end{aligned}$$

Since $\zeta \in U(\Sigma)$ is a product of $N - 2$ commutators by Lemma 8, we are done.

If $\Phi = A_{4k+1}$, we start by writing $g = -u_1 v_1 u_2 v_2$ for some $u_i \in U^+$, $v_i \in U^-$. Then, as previously, $g = -c_2 \varphi \psi$, where φ is similar to

a companion matrix by Lemma 2, while ψ^{-1} is similar to a minus companion matrix by Remark 4. Then for $z_1 = \mu\varphi\mu^{-1}$ and $z_2 = \nu\psi^{-1}\nu^{-1}$ one has $z_1z_2^{-1} = -\zeta$ for some $\zeta \in U(\Sigma)$. Again,

$$g = -c_2\varphi\psi = -c_2 \cdot (-[\mu^{-1}\zeta\nu, \psi^{-1}] \cdot \zeta^{\mu\psi})$$

is a product of N commutators. \square

4. FINAL REMARKS

We first note that starting with a uniriangular factorization of different length, one immediately obtains nice bounds on the commutator width. For example, Chevalley groups over boolean rings admit the unitriangular factorization $E(\Phi) = U^+U^-U^+$ of length 3, thus any of its elements is conjugated to the product uv for some $u \in U^+$, $v \in U^-$. It follows that each element of $E(\Phi, R)$ can be expressed as a product of $N - 1$ commutators (N as in Theorem 1).

Another example is $E(\Phi, \mathbb{Z}[1/p])$, which admits the factorization of length 5 [VSS12, Vse13], thus having the same estimate on its commutator width, as groups over rings of stable rank 1.

In [VW90, AVY95] the commutator width is computed also for what is called the *extended* classical groups (the examples being GL_n , GSp_{2n} , GO_n , etc.). The resulting estimates are slightly better, because one can start with the Gauss decomposition [SSV12]

$$E(\Phi, R) = H(\Phi, R)U^+(\Phi, R)U^-(\Phi, R)U^+(\Phi, R)$$

instead of the unitriangular factorization. Then one modifies Lemma 2 as follows (here $\overline{T}_{\text{sc}}(\Phi)$ is the extended torus, see [BM75, Vav09]):

Lemma 9. *For any $b \in H(\Phi)U^+(\Phi)$ exists $\eta \in \overline{T}_{\text{sc}}(\Phi)U^+(\Phi)$ such that $\eta b \pi \eta^{-1}$ is a companion matrix.*

The result is obtained in the same way as in Theorem 1. Moreover, in this setting there is no need to treat the case $\Phi = \mathbf{A}_{4k+1}$ individually, since one can put $\widehat{w}_0 = p_n$ as in [VW90].

One more interesting thing about [AVY95] is an even better estimate in case of even-dimensional orthogonal group O_{2n} . The trick is to use not the Coxeter element of the Weyl group $W(\mathbf{D}_\ell)$, but rather a certain Coxeter element of the $\mathbf{A}_{\ell-1}$ subsystem Δ_ℓ , composed with an inner automorphism, corresponding to the symmetry of \mathbf{D}_ℓ Dynkin

diagramm. This allows to take $\Sigma = \Sigma_\ell \cap \Delta_{\ell-1}$, acted on by $E(\Delta_\ell \cap \Delta_{\ell-1})$, which is exactly the case of $A_{\ell-1}$. However, this automorphism is inner only for O_{2n} , but not for SO_{2n} , despite what is claimed in [AVY95].

The following argument, showing that this automorphism is inner in O_{2n} , is due to S. Garibaldi.

Let $\rho : G \rightarrow GL(V)$ be an irreducible representation of G with the highest weight λ . Multiplying the given automorphism σ of Φ by an element of the Weyl group, we can assume that $\sigma(\Pi) = \Pi$ (and σ sends dominant weights to dominant weights). We wish to find an $x \in GL(V)$ such that $\sigma(g) = xgx^{-1}$ for every $g \in G$. Proposition 2.2 of [BGL14] says that such x exists if and only if $\sigma(\lambda) = \lambda$. Now let ρ be the natural representation of O_{2n} , and since ϖ_1 is fixed by the symmetry, such x exists in GL_{2n} . For every G -invariant polynomial function f on V , xf is $\rho(G)$ -invariant. But $\rho(G) = G$, so xf is G -invariant. If f is a nondegenerate quadratic form on V , xf is $SO(f)$ -invariant and is a nondegenerate quadratic form, so it must be a scalar multiple of f .

For even-dimensional Spin group no such element exist in Pin or Clifford group, as it must swap the highest weights of two half-spin summand of its spin representation.

REFERENCES

- [AVY95] F. A. Arlinghaus, L. N. Vaserstein, and Hong You, *Commutators in pseudo-orthogonal groups*, J. Austral. Math. Soc. Ser. A **59** (1995), no. 3, 353–365.
- [BGL14] H. Bermudez, S. Garibaldi, and V. Larsen, *Linear preservers and representations with a 1-dimensional ring of invariants*, Trans. Amer. Math. Soc. **366** (2014), no. 6, 4755–4780.
- [BM75] S. Berman and R. Moody, *Extensions of Chevalley groups*, Israel Journal of Mathematics **22** (1975), no. 1, 42–51.
- [BV00] A. Bak and N. Vavilov, *Structure of hyperbolic unitary groups. I. Elementary subgroups*, Algebra Colloq. **7** (2000), no. 2, 159–196.
- [EG98] E. W. Ellers and N. Gordeev, *On the conjectures of J. Thompson and O. Ore*, Trans. Amer. Math. Soc. **350** (1998), no. 9, 3657–3671.
- [LOST10] M. W. Liebeck, E. A. O’Brien, A. Shalev, and P. H. Tiep, *The Ore conjecture*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 4, 939–1008.
- [PSV98] E. Plotkin, A. Semenov, and N. Vavilov, *Visual basic representations: an atlas*, Internat. J. Algebra Comput. **8** (1998), no. 1, 61–95.

- [SSV12] A. Smolensky, B. Sury, and N. Vavilov, *Gauss decomposition for Chevalley groups, revisited*, International Journal of Group Theory **1** (2012), no. 1, 3–16.
- [Vav00] N. Vavilov, *A third look at weight diagrams*, Rend. Sem. Mat. Univ. Padova **104** (2000), 201–250.
- [Vav01] N. A. Vavilov, *Do it yourself structure constants for Lie algebras of types E_l* , Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **281** (2001), no. Vopr. Teor. Predst. Algebr. i Grupp. 8, 60–104, 281.
- [Vav09] N. Vavilov, *Weight elements of Chevalley groups*, St. Petersburg Mathematical Journal **20** (2009), no. 1, 23–57.
- [Vse13] M. Vsemirnov, *Short unitriangular factorizations of $SL_2(\mathbb{Z}[1/p])$* , The Quarterly Journal of Mathematics (2013), has044.
- [VSS12] N. A. Vavilov, A. V. Smolensky, and B. Sury, *Unitriangular factorizations of Chevalley groups*, Journal of Mathematical Sciences **183** (2012), no. 5, 584–599.
- [VW90] L. N. Vaserstein and E. Wheland, *Commutators and companion matrices over rings of stable rank 1*, Linear Algebra Appl. **142** (1990), 263–277.

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